Orbit Determination Using Analytic Partial Derivatives of Perturbed Motion

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State transition matrices for the motion of space vehicles have been calculated both numerically and analytically. The former method, for example, integrates the "variational equations." It yields accurate results when the same force model is used in the variational equations as in the equations of motion. The main advantage of the latter method, i.e., using analytic partial derivatives, is faster computation, but, if the transition is made by Keplerian formulas to a time many revolutions from epoch, the accuracy of the matrix is severely degraded. It is shown in this paper that the state transition matrix of a satellite orbit may be calculated analytically including the effects of perturbations. This is accomplished by generating each perturbation in the Gaussian form (using three mutually perpendicular components of the perturbing acceleration) to find the transition matrix from epoch state to the state at any other time. This analytic method is considerably faster than the numerical integration of the variational equations and it is shown that the methods are in good agreement. The state vector considered includes orbit parameters, which are not singular for any elliptic orbits except retrograde equatorial, and the parameters associated with the perturbations. These perturbations include atmospheric drag, thrust periods, impulses, direct solar radiation pressure, oxidizer outgassing, and the geopotential. The method, however, can be extended to any perturbation, nor is it restricted to geocentric orbit calculations. Results are presented in tabular form. Partial derivatives of the satellite's position with respect to the orbital elements and other vehicular parameters, such as the ballistic parameter C_DA/m, are listed in the tables as obtained by each of three methods: by the analytical procedure, by integrating the variational equations, and by differencing neighboring trajectories. The comparisons are made on a satellite with a period of 90 min, over complete revolutions one day after epoch and one week after epoch.

Nomenclature†

A = effective cross-sectional area of satellite to atmospheric drag
B = ballistic parameter of a satellite; $B = C_DA/m$
C_D = drag coefficient (dimensionless)
E = eccentric anomaly
J_2 = coefficient of the second zonal harmonic in the earth's gravitational potential
L = mean longitude; $L = M + \Omega + \omega = M + \pi = L_0 + n(t - t_0)$
M = mean anomaly
a = mean distance, or semimajor axis of elliptical orbit
a_e = mean equatorial radius of the earth
a_f = equinoctial element; $a_f = e \cos\pi$
a_g = equinoctial element; $a_g = e \sin\pi$
e = eccentricity of orbit
f = true anomaly
$\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{w}$ = $\boldsymbol{f}$ is a unit vector in the orbit plane directed at an angle Ω in the retrograde direction from the ascending node; $\boldsymbol{g}$ is 90° from $\boldsymbol{f}$, measured in the orbit plane in the direction of motion; $\boldsymbol{w}$ completes the right-handed orthonormal set with $\boldsymbol{f},\boldsymbol{g}$. The set $\boldsymbol{f},\boldsymbol{g},\boldsymbol{w}$ is called the equinoctial set of reference axes
i = inclination of the orbit plane with respect to the equatorial plane of the earth
l = true longitude; $l = f + \Omega + \omega = f + \pi$
m = mass
n = mean angular motion
p = semiparameter or semilatus rectum; $p = a(1 - e^2)$
p_j,p_k = arbitrary correction parameters
$\mathbf{r}$ = radius vector from the center of the earth to the satellite
$\dot{\mathbf{r}}$ = derivative of the radius vector with respect to time
r = radial distance from the center of the earth to the satellite; $r = |\mathbf{r}|$
$\dot{r}$ = radial component of satellite's velocity vector; $\dot{r} = \boldsymbol{u}\cdot\dot{\mathbf{r}}$
t = time; when unsubscripted t means current time; subscripts are used when any other time is referenced
$\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ = $\boldsymbol{u}$ is a unit vector along the radius vector $\mathbf{r}$; $\boldsymbol{u} = \mathbf{r}/r$; $\boldsymbol{v}$ is a unit vector 90° from $\boldsymbol{u}$ in the orbit plane, measured in the direction of motion; $\boldsymbol{w}$ is a unit vector normal to the orbit plane, and completes the right-handed orthonormal set with $\boldsymbol{u},\boldsymbol{v}$
v_a = speed of the satellite relative to the earth's atmosphere; $v_a = |\dot{\mathbf{r}}_a|$
x,y,z = components of the position vector $\mathbf{r}$ referred to the inertial reference axes
z = argument of Bessel function
Δ = difference operator
μ = product of the mass of the earth times the universal constant of gravitation
π = longitude of perigee; $\pi = \Omega + \omega$
ρ = atmospheric density at satellite's altitude
φ = null matrix
χ,ψ = equinoctial parameters for orbit-plane orientation; $\chi = \tan(i/2)\sin\Omega$; $\psi = \tan(i/2)\cos\Omega$
ω = argument of perigee
Ω = longitude of the ascending node
o = subscript denoting epoch time
m = superscript denoting model parameters

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† Boldface italic denotes unit vector (this notation is not used on components or derivatives of unit vectors).

o = superscript denoting orbit parameters
$\cdot$ = derivative with respect to time

I. Introduction

THE primary purpose of this paper is to demonstrate the value of using analytic coefficients or "partials" in the differential correction of an orbit. The major departure from earlier theories[1,2] is that these analytic partials include the effects of perturbations. These partials contribute to dramatic savings in computer time and to significant increases in accuracy.

A change in the orbital elements and model parameters at the epoch, Δp_{k0}, is related to a change in position $\Delta\mathbf{r}$ and velocity $\Delta\dot{\mathbf{r}}$ at time t as follows:

$$\Delta\mathbf{r} = \sum \frac{\partial \mathbf{r}}{\partial p_{k0}} \Delta p_{k0} = \sum_j \frac{\partial \mathbf{r}}{\partial p_j} \sum \frac{\partial p_j}{\partial p_{k0}} \Delta p_{k0}$$
$$\Delta\dot{\mathbf{r}} = \sum_j \frac{\partial \dot{\mathbf{r}}}{\partial p_j} \sum \frac{\partial p_j}{\partial p_{k0}} \Delta p_{k0} \tag{1}$$

The partial derivatives of $\mathbf{r}$ and $\dot{\mathbf{r}}$ with respect to the elements p_j at time t represent the two-body or Keplerian terms. They are obtained by analytic differentiation of the two-body representation formulas. The partial derivatives of the orbital elements p_j at time t with respect to the p_{k0} at the epoch time are due to the perturbations acting on the satellite, such as the earth's bulge, atmospheric drag, etc.

The partial derivatives can be developed in a number of ways. A mathematically rigorous method of obtaining the partials $\partial\mathbf{r}/\partial p_{k0}$ and $\partial\dot{\mathbf{r}}/\partial p_{k0}$ is by means of the variational equations. These equations[3] are integrated numerically along with the equations of motion in rectangular component form. They include the two-body acceleration in addition to the effects of the perturbing accelerations. The partials obtained in whis way include all of the secular and periodic variations caused by the forces considered to act on the satellite. However, it has been demonstrated[4] that neglect of even small perturbative accelerations in the variational equations can lead very quickly to large errors in the partial derivatives. This is due to the initialization difficulty that exists with the variational equations rather than the periodic contributions made by these perturbative accelerations.

Completely analytic partials are the most efficient for computation purposes. However, they are also the most cumbersome to derive and manipulate. These partials are obtained by differentiating both the two-body and perturbation portions of the representation formulas. Early applications of analytic partials assumed that the differential perturbation terms were of order $(\Delta p)^2$ and could be neglected along with the other second-order terms of Eq. (1). This assumption is valid for short observation spans but not when the observations are separated from epoch by, say, 20 or more revolutions. A change in one of the epoch parameters will not only affect the later value of that parameter, but of all the other parameters as well. It does this because this change in the orbit alters the perturbative accelerations on the satellite. The perturbative accelerations in turn influence all parameters of the orbit. Figure 1 compares analytic partial derivatives including perturbations with analytic Keplerian partials, and shows the circumferential component $n\boldsymbol{v}\cdot(\partial\mathbf{r}/\partial n)$ with and without the effect of drag for a typical low-altitude satellite.

Expressions for the radial, circumferential, and binormal components of Eq. (1) are obtained by taking the dot products with the unit vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$, respectively. The resulting set of equations is written in matrix form as

$$\Delta\mathbf{X} = G_1 G_2 \Delta\mathbf{p}_{k0} \tag{2}$$

where

$$\Delta\mathbf{X} = \begin{bmatrix} \boldsymbol{u}\cdot\Delta\mathbf{r} \\ \boldsymbol{v}\cdot\Delta\mathbf{r} \\ \boldsymbol{w}\cdot\Delta\mathbf{r} \\ \boldsymbol{u}\cdot\Delta\dot{\mathbf{r}} \\ \boldsymbol{v}\cdot\Delta\dot{\mathbf{r}} \\ \boldsymbol{w}\cdot\Delta\dot{\mathbf{r}} \end{bmatrix}$$

is a column vector consisting of the radial, circumferential, and binormal components of the position and velocity displacements at time t; G_1 is a matrix representing the partial derivatives of the radial, circumferential, and binormal components of position and velocity with respect to the orbital elements at time t (the Keplerian partials); G_2 is a matrix representing the partial derivatives of the orbital elements at time t with respect to the orbital elements and model parameters at the epoch time. The first six columns are assigned to the orbital elements. That 6×6 matrix is called G_2^0. The remaining columns, called G_2^m, are assigned to the model parameters. Thus G_2 is partitioned as $G_2 = G_2^0 \vdots G_2^m$. $\Delta\mathbf{p}_{k0}$ is a column vector representing the corrections to the elements and model parameters at epoch time.

The G_1 and G_2 matrices are developed in terms of a set of elements that is free from both low-eccentricity and low-inclination difficulties. The set, which is referred to as the equinoctial set, consists of a_f, a_g, n, L, χ, and ψ.

II. Development of Analytic Partials

Keplerian Analytic Partials

The G_1 matrix [see Eq. (2)] is defined as follows in terms of the equinoctial set of elements:

$$G_1 = \begin{bmatrix}
\boldsymbol{u}\cdot\frac{\partial\mathbf{r}}{\partial a_f} & \boldsymbol{u}\cdot\frac{\partial\mathbf{r}}{\partial a_g} & \boldsymbol{u}\cdot n\frac{\partial\mathbf{r}}{\partial n} & \boldsymbol{u}\cdot\frac{\partial\mathbf{r}}{\partial L} & \boldsymbol{u}\cdot\frac{\partial\mathbf{r}}{\partial\chi} & \boldsymbol{u}\cdot\frac{\partial\mathbf{r}}{\partial\psi} \\
\boldsymbol{v}\cdot\frac{\partial\mathbf{r}}{\partial a_f} & \boldsymbol{v}\cdot\frac{\partial\mathbf{r}}{\partial a_g} & \boldsymbol{v}\cdot n\frac{\partial\mathbf{r}}{\partial n} & \boldsymbol{v}\cdot\frac{\partial\mathbf{r}}{\partial L} & \boldsymbol{v}\cdot\frac{\partial\mathbf{r}}{\partial\chi} & \boldsymbol{v}\cdot\frac{\partial\mathbf{r}}{\partial\psi} \\
\boldsymbol{w}\cdot\frac{\partial\mathbf{r}}{\partial a_f} & \boldsymbol{w}\cdot\frac{\partial\mathbf{r}}{\partial a_g} & \boldsymbol{w}\cdot n\frac{\partial\mathbf{r}}{\partial n} & \boldsymbol{w}\cdot\frac{\partial\mathbf{r}}{\partial L} & \boldsymbol{w}\cdot\frac{\partial\mathbf{r}}{\partial\chi} & \boldsymbol{w}\cdot\frac{\partial\mathbf{r}}{\partial\psi} \\
\boldsymbol{u}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial a_f} & \boldsymbol{u}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial a_g} & \boldsymbol{u}\cdot n\frac{\partial\dot{\mathbf{r}}}{\partial n} & \boldsymbol{u}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial L} & \boldsymbol{u}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial\chi} & \boldsymbol{u}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial\psi} \\
\boldsymbol{v}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial a_f} & \boldsymbol{v}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial a_g} & \boldsymbol{v}\cdot n\frac{\partial\dot{\mathbf{r}}}{\partial n} & \boldsymbol{v}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial L} & \boldsymbol{v}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial\chi} & \boldsymbol{v}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial\psi} \\
\boldsymbol{w}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial a_f} & \boldsymbol{w}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial a_g} & \boldsymbol{w}\cdot n\frac{\partial\dot{\mathbf{r}}}{\partial n} & \boldsymbol{w}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial L} & \boldsymbol{w}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial\chi} & \boldsymbol{w}\cdot\frac{\partial\dot{\mathbf{r}}}{\partial\psi}
\end{bmatrix}$$

The top three rows of the 6×6 G_1 matrix represent the position partial derivatives; the lower three rows represent the velocity partials. This matrix relates orbit position and velocity at an observation time, resolved into the $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ directions, to the osculating equinoctial parameters at that time.

Two-body formulas required for the derivation of the G_1 partials include

$$E - e\sin E = M = L - \pi \text{ (M is mean anomaly)} \tag{3}$$

$$r = a(1 - e\cos E) \tag{4}$$

$$r\cos f = a(\cos E - e) \text{ (f is true anomaly)}$$

$$r\sin f = a(1 - e^2)^{1/2}\sin E \text{ (E is eccentric anomaly)}$$

The satellite position vector is $\mathbf{r} = r\boldsymbol{u}$; differentiation leads to $\Delta\mathbf{r} = \Delta r\,\boldsymbol{u} + r\,\Delta\boldsymbol{u}$. The radial component of this displacement is $\boldsymbol{u}\cdot\Delta\mathbf{r} = \Delta r$.

Evaluation of the position partials is similar to the derivation in Ref. 1. The results are summarized for the radial

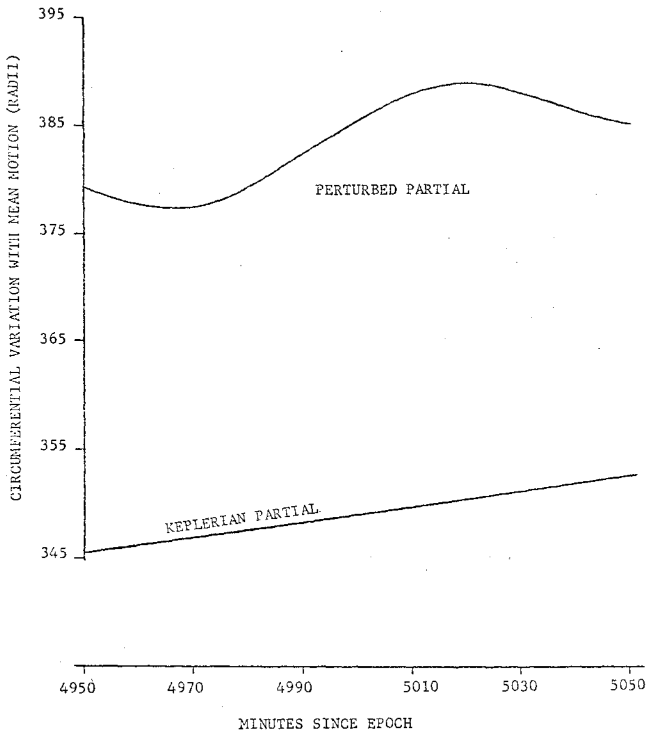

Fig. 1 Comparison of analytic partials n v $\partial r/\partial n$.

components;

$$\Delta r = R_f \Delta a_f + R_g \Delta a_g + R_n \Delta n/n + R_L \Delta L \qquad (5)$$

where

$$\boldsymbol{u}\cdot\partial \mathbf{r}/\partial a_f = R_f = (a^2/r)[a_f - \cos(E + \pi)]$$

$$\boldsymbol{u}\cdot\partial \mathbf{r}/\partial a_g = R_g = (a^2/r)[a_g - \sin(E + \pi)]$$

$$n\boldsymbol{u}\cdot\partial \mathbf{r}/\partial n = R_n = -\tfrac{2}{3}r$$

$$\boldsymbol{u}\cdot\partial \mathbf{r}/\partial L = R_L = (a^2/r)[e \sin E]$$

$$\boldsymbol{u}\cdot\partial \mathbf{r}/\partial \chi = 0,\ \boldsymbol{u}\cdot\partial \mathbf{r}/\partial \psi = 0$$

The circumferential and binormal components of the position partials are

$$\boldsymbol{v}\cdot\partial \mathbf{r}/\partial a_f = U_f,\ \boldsymbol{v}\cdot\partial \mathbf{r}/\partial L = U_L$$

$$\boldsymbol{w}\cdot\partial \mathbf{r}/\partial a_f = 0,\ \boldsymbol{w}\cdot\partial \mathbf{r}/\partial L = 0$$

$$\boldsymbol{v}\cdot\partial \mathbf{r}/\partial a_g = U_g,\ \boldsymbol{v}\cdot\partial \mathbf{r}/\partial \chi = U_\chi$$

$$\boldsymbol{w}\cdot\partial \mathbf{r}/\partial a_g = 0,\ \boldsymbol{w}\cdot\partial \mathbf{r}/\partial \chi = B_\chi$$

$$\boldsymbol{v}\cdot n\partial \mathbf{r}/\partial n = 0,\ \boldsymbol{v}\cdot\partial \mathbf{r}/\partial \psi = U_\psi$$

$$\boldsymbol{w}\cdot n\partial \mathbf{r}/\partial n = 0,\ \boldsymbol{w}\cdot\partial \mathbf{r}/\partial \psi = B_\psi$$

where

$$U_f = \frac{a^2}{r}\left\{\left(1 + \frac{r}{a}\right)\sin(E + \pi) + a_f e \sin E \times \frac{e^2 - [1 + (1 - e^2)^{1/2}]e \cos E}{(1 - e^2)^{1/2}[1 + (1 - e^2)^{1/2}]^2} - \frac{a_g}{1 + (1 - e^2)^{1/2}}\right\}$$

$$U_g = \frac{a^2}{r}\left\{-\left(1 + \frac{r}{a}\right)\cos(E + \pi) + a_g e \sin E \times \frac{e^2 - [1 + (1 - e^2)^{1/2}]e \cos E}{(1 - e^2)^{1/2}[1 + (1 - e^2)^{1/2}]^2} + \frac{a_f}{1 + (1 - e^2)^{1/2}}\right\}$$

$$U_L = (a^2/r)(1 - e^2)^{1/2},\ U_\chi = rw_y,\ U_\psi = rw_x$$

$$B_\chi = -r(1 + w_z)\cos l,\ B_\psi = r(1 + w_z)\sin l$$

The partials relating satellite velocity to the equinoctial orbit parameters at a given time, as conveyed in the G_1 matrix are

$$\boldsymbol{u}\cdot\partial \dot{\mathbf{r}}/\partial a_f = \dot{R}_f - \dot{f}U_f,\ \boldsymbol{v}\cdot\partial \dot{\mathbf{r}}/\partial a_f = \dot{U}_f + (\dot{r}/r)U_f$$

$$\boldsymbol{w}\cdot\partial \dot{\mathbf{r}}/\partial a_f = 0$$

$$\boldsymbol{u}\cdot\partial \dot{\mathbf{r}}/\partial a_g = \dot{R}_g - \dot{f}U_g,\ \boldsymbol{v}\cdot\partial \dot{\mathbf{r}}/\partial a_g = \dot{U}_g + (\dot{r}/r)U_g$$

$$\boldsymbol{w}\cdot\partial \dot{\mathbf{r}}/\partial a_g = 0$$

$$\boldsymbol{u}\cdot n\partial \dot{\mathbf{r}}/\partial n = \dot{R}_n,\ \boldsymbol{v}\cdot n\partial \dot{\mathbf{r}}/\partial n = U_n,\ \boldsymbol{w}\cdot n\partial \dot{\mathbf{r}}/\partial n = 0$$

$$\boldsymbol{u}\cdot\partial \dot{\mathbf{r}}/\partial L = \dot{R}_L - \dot{f}U_L,\ \boldsymbol{v}\cdot\partial \dot{\mathbf{r}}/\partial L = 0,\ \boldsymbol{w}\cdot\partial \dot{\mathbf{r}}/\partial L = 0$$

$$\boldsymbol{u}\cdot\partial \dot{\mathbf{r}}/\partial \chi = -\dot{f}U_\chi,\ \boldsymbol{v}\cdot\partial \dot{\mathbf{r}}/\partial \chi = (\dot{r}/r)U_\chi$$

$$\boldsymbol{w}\cdot\partial \dot{\mathbf{r}}/\partial \chi = (\dot{r}/r)B_\chi + \dot{f}B_\psi$$

$$\boldsymbol{u}\cdot\partial \dot{\mathbf{r}}/\partial \psi = -\dot{f}U_\psi,\ \boldsymbol{v}\cdot\partial \dot{\mathbf{r}}/\partial \psi = (\dot{r}/r)U_\psi$$

$$\boldsymbol{w}\cdot\partial \dot{\mathbf{r}}/\partial \psi = (\dot{r}/r)B_\psi - \dot{f}B_\chi$$

where

$$\dot{R}_f = \left(\frac{\mu}{a}\right)^{1/2}\left(\frac{a}{r}\right)^3 [\sin(E + \pi) - a_f e \sin E - a_g]$$

$$\dot{R}_g = \left(\frac{\mu}{a}\right)^{1/2}\left(\frac{a}{r}\right)^3 [-\cos(E + \pi) - a_g e \sin E + a_f]$$

$$\dot{R}_n = \frac{\dot{r}}{3},\ \dot{R}_L = \left(\frac{\mu}{a}\right)^{1/2}\left(\frac{a}{r}\right)^3 (e \cos E - e^2)$$

$$\dot{U}_f = (\mu p)^{1/2}\frac{a^2}{r^3}\left[\cos(E + \pi) - a_f\left(1 + \frac{r^2}{ap}\right)\right]$$

$$\dot{U}_g = (\mu p)^{1/2}\frac{a^2}{r^3}\left[\sin(E + \pi) - a_g\left(1 + \frac{r^2}{ap}\right)\right]$$

$$\dot{U}_n = \frac{r\dot{f}}{3},\ \dot{U}_L = -(\mu p)^{1/2}\frac{a^2}{r^3} e \sin E$$

Inclusion of Perturbations in Analytic Partials

The G_2^0 portion of the G_2 matrix [Eq. (2)] contains the partial derivatives of the current orbital elements with respect to the orbital elements at the latest epoch time.‡ These partials are due to the perturbations acting on the satellite,

$$G_2^0 = I' + G_2^0(\text{bulge}) + G_2^0(\text{drag}) + \ldots \qquad (6)$$

where I' is an identity matrix plus the Keplerian term $n_0(t - t_0)$ in the matrix element $n_0\, \partial L/\partial n_0$ and n_0/n for the element $(n_0/n)\ (\partial n/\partial n_0)$. The contribution made to the G_2^0 matrix by the earth's bulge, atmospheric drag, duration thrust, impulsive thrust, outgassing, and direct solar radiation pressure have been developed and are presented in Ref. 5. Only the analytic partials due to the earth's bulge and atmospheric drag are developed in this paper. They serve to illustrate the techniques used.

Earth's bulge

The partials are fully analytic and complete for the secular perturbations due to the earth's second zonal harmonic J_2. It has been established that periodic terms due to J_2, and secular and periodic terms due to the remaining zonal and tesseral harmonics of the geopotential, are not required in the analytic partials (G_2^0 matrix).

‡ The "latest epoch" is the time of the correction epoch or the time of the last discontinuity, such as the beginning or end of thrust.

Atmospheric drag

The terms $\int \delta a/\delta t$ and $\int (1/e)\ (\delta e/\delta t)$ caused by drag are complete; the integration is done numerically with a low-order integration method. These complete integrals are then used in otherwise fully analytic partials. The partials are based on secular variations in a and e from an analytic theory good to the first order of eccentricity.

Earth's Bulge

The partial derivatives for the $G_2{}^0$ matrix, due to the earth's second zonal harmonic J_2, are based upon general perturbations expressions that include the secular variations in Ω, ω, M;

$$\left.\begin{aligned}\dot{\Omega}_0 &= -\frac{3}{2}J_2\left(\frac{a_e}{p_0}\right)^2 n_0 \cos i_0\\ \dot{\pi}_0 = \dot{\Omega}_0 + \dot{\omega}_0 &= \frac{3}{2}J_2\left(\frac{a_e}{p_0}\right)^2 \times\\ &\qquad n_0\left(2 - \frac{5}{2}\sin^2 i_0 - \cos i_0\right)\\ \dot{L}_0 = \dot{\pi}_0 + \dot{M}_0 = \dot{\pi}_0 &+ \frac{3}{2}J_2\left(\frac{a_e}{p_0}\right)^2 \times\\ &\qquad (1 - e_0{}^2)^{1/2} n_0\left(1 - \frac{3}{2}\sin^2 i_0\right)\end{aligned}\right\} \quad (7)$$

The subscript 0 indicates that the quantities are evaluated at the epoch t_0. The complete collection of partials follows. Note that these fully analytic partials include the secular variations due to J_2 only and that there has been no truncation of the orbital eccentricity. The quantity $(\partial\dot{\pi}_0/\partial i_0)/\sin i_0$, required in the partials, is, from Eq. (7),

$$\frac{1}{\sin i_0}\frac{\partial\dot{\pi}_0}{\partial i_0} = \frac{3}{2}J_2\left(\frac{a_e}{p_0}\right)^2 n_0(1 - 5\cos i_0) \qquad (8)$$

Partials of element a_f:

$$\frac{\partial a_f}{\partial a_{f0}} = \cos[\dot{\pi}_0(t - t_0)] - 1 - 4a_{f0}a_g\frac{a_0}{p_0}\dot{\pi}_0(t - t_0)$$

$$\frac{\partial a_f}{\partial a_{g0}} = -\sin[\dot{\pi}_0(t - t_0)] - 4a_{g0}a_g\frac{a_0}{p_0}\dot{\pi}_0(t - t_0)$$

$$n_0\frac{\partial a_f}{\partial n_0} = -\frac{7}{3}a_g\pi_0(t - t_0),\ \frac{\partial a_f}{\partial L_0} = 0$$

$$\frac{\partial a_f}{\partial \chi_0} = -a_g\chi_0(1 + w_{z0})^2\frac{1}{\sin i_0}\frac{\partial\dot{\pi}_0}{\partial i_0}(t - t_0)$$

$$\frac{\partial a_f}{\partial\psi_0} = \frac{\psi_0}{\chi_0}\frac{\partial a_f}{\partial\chi_0}$$

Partials of element a_g:

$$\frac{\partial a_g}{\partial a_{f0}} = \sin[\dot{\pi}_0(t - t_0)] + 4a_{f0}a_f\frac{a_0}{p_0}\dot{\pi}_0(t - t_0)$$

$$\frac{\partial a_g}{\partial a_{g0}} = \cos[\dot{\pi}_0(t - t_0)] - 1 + 4a_{g0}a_f\frac{a_0}{p_0}\dot{\pi}_0(t - t_0)$$

$$n_0\frac{\partial a_g}{\partial n_0} = \frac{7}{3}a_f\dot{\pi}_0(t - t_0),\ \frac{\partial a_g}{\partial L_0} = 0$$

$$\frac{\partial a_g}{\partial\chi_0} = -\frac{a_f}{a_g}\frac{\partial a_g}{\partial\chi_0},\ \frac{\partial a_g}{\partial\psi_0} = -\frac{a_f}{a_g}\frac{\partial a_f}{\partial\psi_0}$$

Partials of angular mean motion n:

$$\frac{\partial n}{\partial a_{f0}} = \frac{\partial n}{\partial a_{g0}} = n_0\frac{\partial n}{\partial n_0} = \frac{\partial n}{\partial L_0} = \frac{\partial n}{\partial\chi_0} = \frac{\partial n}{\partial\psi_0} = 0$$

Partials of mean longitude L:

$$\frac{\partial L}{\partial a_{f0}} = a_{f0}\frac{a_0}{p_0}(3\dot{L}_0 + \dot{\pi}_0)(t - t_0),\ \frac{\partial L}{\partial a_{g0}} = \frac{a_{g0}}{a_{f0}}\frac{\partial L}{\partial a_{f0}}$$

$$n_0\frac{\partial L}{\partial n_0} = \frac{7}{3}\dot{L}_0(t - t_0),\ \frac{\partial L}{\partial L_0} = 0$$

$$\frac{\partial L}{\partial\chi_0} = \chi_0(1 + w_{z0})^2 \times \left\{[5 + 3(1 - e_0{}^2)^{1/2}]\dot{\Omega}_0 - \frac{1}{\cos i_0}\dot{\Omega}_0\right\}(t - t_0)$$

$$\frac{\partial L}{\partial\psi_0} = \frac{\psi_0}{\chi_0}\frac{\partial L}{\partial\chi_0}$$

Partials of element χ:

$$\frac{\partial\chi}{\partial a_{f0}} = 4\psi a_{f0}\frac{a_0}{p_0}\dot{\Omega}_0(t - t_0),\ \frac{\partial\chi}{\partial a_{g0}} = \frac{a_{g0}}{a_{f0}}\frac{\partial\chi}{\partial a_{f0}}$$

$$n_0\frac{\partial\chi}{\partial n_0} = \frac{7}{3}\psi\dot{\Omega}_0(t - t_0),\ \frac{\partial\chi}{\partial L_0} = 0$$

$$\frac{\partial\chi}{\partial\chi_0} = \cos[\dot{\Omega}_0(t - t_0)] - 1 - \psi\chi_0(1 + w_{z0})^2\frac{1}{\cos i_0}\dot{\Omega}_0(t - t_0)$$

$$\frac{\partial\chi}{\partial\psi_0} = \sin[\dot{\Omega}_0(t - t_0)] - \psi\psi_0(1 + w_{z0})^2\frac{1}{\cos i_0}\dot{\Omega}_0(t - t_0)$$

Partials of element ψ:

$$\frac{\partial\psi}{\partial a_{f0}} = -4\chi a_{f0}\frac{a_0}{p_0}\dot{\Omega}_0(t - t_0),\ \frac{\partial\psi}{\partial a_{g0}} = \frac{a_{g0}}{a_{f0}}\frac{\partial\psi}{\partial a_{f0}}$$

$$n_0\frac{\partial\psi}{\partial n_0} = -\frac{7}{3}\chi\dot{\Omega}_0(t - t_0),\ \frac{\partial\psi}{\partial L_0} = 0$$

$$\frac{\partial\psi}{\partial\chi_0} = -\sin[\dot{\Omega}_0(t - t_0)] + \chi\chi_0(1 + w_{z0})^2\frac{1}{\cos i_0}\dot{\Omega}_0(t - t_0)$$

$$\frac{\partial\psi}{\partial\psi_0} = \cos[\dot{\Omega}_0(t - t_0)] - 1 + \chi\psi_0(1 + w_{z0})^2\frac{1}{\cos i_0}\dot{\Omega}_0(t - t_0)$$

Atmospheric Drag

The $G_2{}^0$ matrix of partial derivatives for atmospheric drag is based on general perturbations expressions[6]§ which include the secular variations in semimajor axis and eccentricity (to zero-order eccentricity). By assuming a static spherical atmosphere in which the density decays exponentially with altitude (at least in the spherical shell traversed by the satellite),

$$\left.\begin{aligned}\frac{\delta a}{\delta t} &= -B\rho_\pi na^2\exp(-z)B_0(z)\\ \frac{1}{e}\frac{\delta e}{\delta t} &= -B\rho_\pi na\exp(-z)\frac{ka}{2}B_1(z)\end{aligned}\right\} \quad (9)$$

where k is the reciprocal scale height, $k = -(1/\rho)d\rho/dH$, H is the height above mean sea level, and ρ is the atmospheric density at height H; $z = kae$, ρ_π = density at perigee, $\rho_\pi = \rho_0\exp\{-k[a(1 - e) - r_0]\}$, where ρ_0 is a reference atmospheric density at some geocentric distance r_0 near perigee, $B_0(z)$, $B_1(z), \ldots,$ are related to the Bessel functions $I_0(z), I_1(z), \ldots,$ by

§ The theory developed in Ref. 6 includes the altitude variations arising from motion around the oblate earth and the eccentricity of the orbit, but these terms were found to be unnecessary in the partials for the differential correction.

Table 1 Accuracy of partials in parts per 600 after one day[a]

Element:	a_f	a_g	n	L	χ	ψ	
Earth's bulge[b]							
u	A[a]	$1/A$	A	$1/A$	A	A	
v	A	$2/A$	A	$3/A$	$1/A$	$1/A$	
w	A	A	A	A	A	A	
Atmospheric drag[b]							
u	1	3	A	A	A	A	
v	[c]	[c]	A	1/1	$A/1$	A	
w	A	A	A	A	A	A	
Radial thrust							
			All A				
Circumferential thrust							
u	A	A	A	A	A	A	
v	1	4	A	A	A	A	
w	A	A	A	A	A	A	
Binormal thrust							
			All A				
Radial thrusting rate of change, $\dot{T}_u$							
u	A	A	A	A	A	A	
v	3	A	A	A	A	A	
w	A	A	A	A	A	A	
Circumferential thrusting rate of change, $\dot{T}_v$							
u	A	A	A	A	A	A	
v	1	4	A	A	A	A	
w	A	A	A	A	A	A	
Binormal thrusting rate of change, $\dot{T}_w$							
			All A				
Constants:	B	T_u	T_v	T_w	$\dot{T}_u$	$\dot{T}_v$	$\dot{T}_w$
u	A	3	1	A	2	A	A
v	$^{11}/_4$	[d]	A	A	[e]	A	A
w	A	A	A	2	A	A	2

[a] Implies the accuracy is within the acceptable accuracy range of 1 part in 600, noted if otherwise.
[b] Comparison of analytic/variational equations with numeric partials.
[c] Application of quantitative design goal on these partials is questioned, due to demonstrated nonlinearity of drag interaction between a_f and a_g and mean motion.
[d] Application of quantitative design goal on these partials is questioned due to demonstrated nonlinearity of radial thrust interaction (through eccentricity) to mean motion.
[e] Application of quantitative goal on these partials is questioned, due to nonlinear relationship of radial thrust rate parameter (through eccentricity) to mean motion.

$$B_l(z) = (2/z)^l I_l(z) \quad (l = 0,1,2,\ldots)$$

For $z \leq 2$, which corresponds approximately to $e \leq 0.01$, the $B_l(z)$ are given by

$$B_l(z) = \sum_{r=0}^{\infty} \frac{(z/2)^{2r}}{r!(r+l)!} \quad (l = 0,1,2)$$

For $z > 2$, asymptotic expansions of $B_l(z)$ are used.

Numerical studies of the partials for drag revealed that the analytic approximations

$$\int_{t_0}^{t} \frac{1}{e}\frac{\delta e}{\delta t}\,dt = \left(\frac{1}{e}\frac{\delta e}{\delta t}\right)_0 (t - t_0)$$

and

$$\int_{t_0}^{t} \frac{\delta a}{\delta t}\,dt = \left(\frac{\delta a}{\delta t}\right)_0 (t - t_0)$$

would be inadequate. Instead, the variations $(1/e)(\delta e/\delta t)$ and $(\delta a/\delta t)$ are computed during the ephemeris integration directly from the acceleration due to atmospheric drag. For example,

$$\delta a/\delta t = B\rho v_a (a^2/\mu)\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}_a$$

The variations $(1/e)(\delta e/\delta t)$ and $(\delta a/\delta t)$ are then integrated numerically using a low-order (trapezoidal rule) integration. Note that even though the analytic partials are derived from a first-order eccentricity theory for $(\delta a/\delta t)$ and $(1/e)(\delta e/\delta t)$ [Eq. (9)], the integrals $\int(\delta a/\delta t)dt$ and $\int(1/e)(\delta e/\delta t)dt$ are complete as used in the partials since they are obtained by numerical integration. Strictly speaking, the double integral $\iint(\delta a/\delta t)dt^2$ involved in $\partial L/\partial p_{k0}$ should be performed by a second numerical integration. However, numerical experimentation revealed that this double integration could be approximated satisfactorily by $\frac{1}{2}(t - t_0)\int(\delta a/\delta t)dt$.

The complete collection of partials for drag follows. For convenience in writing the following equations, the auxiliary quantity β is defined as

$$\beta = \frac{k_0{}^2 a_0{}^2}{2}\frac{B_2(z_0)}{B_1(z_0)}\int_{t_0}^{t}\frac{1}{e}\frac{\delta e}{\delta t}\,dt$$

Partials of element a_f:

$$\frac{\partial a_f}{\partial a_{f0}} = \beta a_{f0}{}^2 + \int_{t_0}^{t}\frac{1}{e}\frac{\delta e}{\delta t}\,dt, \quad \frac{\partial a_f}{\partial a_{g0}} = \beta a_{f0}a_{g0}$$

$$n_0\frac{\partial a_f}{\partial n_0} = -\frac{2}{3}a_0 a_{f0}\left\{\frac{1}{2a_0} - k_0\left[1 - e_0\frac{z_0}{2}\frac{B_2(z_0)}{B_1(z_0)}\right]\right\}\int_{t_0}^{t}\frac{1}{e}\frac{\delta e}{\delta t}\,dt$$

$$\frac{\partial a_f}{\partial L_0} = \frac{\partial a_f}{\partial \chi_0} = \frac{\partial a_f}{\partial \psi_0} = 0$$

Partials of element a_g:

$$\frac{\partial a_g}{\partial a_{f0}} = \frac{\partial a_f}{\partial a_{g0}}, \quad \frac{\partial a_g}{\partial a_{g0}} = \beta a_{g0}{}^2 + \int_{t_0}^{t}\frac{1}{e}\frac{\delta e}{\delta t}\,dt$$

$$n_0\frac{\partial a_g}{\partial n_0} = \frac{a_{g0}}{a_{f0}}n_0\frac{\partial a_f}{\partial n_0}, \quad \frac{\partial a_g}{\partial L_0} = \frac{\partial a_g}{\partial \chi_0} = \frac{\partial a_g}{\partial \psi_0} = 0$$

Partials of mean motion n:

$$\frac{1}{n}\frac{\partial n}{\partial a_{f0}} = -\frac{3}{4}a_{f0}a_0k_0{}^2\frac{B_1(z_0)}{B_0(z_0)}\int_{t_0}^{t}\frac{\delta a}{\delta t}\,dt$$

$$\frac{1}{n}\frac{\partial n}{\partial a_{g0}} = \frac{a_{g0}}{a_{f0}}\frac{1}{n}\frac{\partial n}{\partial a_{f0}}$$

$$\frac{n_0}{n}\frac{\partial n}{\partial n_0} = -\frac{n_0}{n}\left\{\frac{2}{a_0} + k_0\left[1 - e_0\frac{z_0}{2}\frac{B_1(z_0)}{B_0(z_0)}\right]\right\}\int_{t_0}^{t}\frac{\delta a}{\delta t}\,dt$$

$$\frac{\partial n}{\partial L_0} = \frac{\partial n}{\partial \chi_0} = \frac{\partial n}{\partial \psi_0} = 0$$

Partials of mean longitude L:

$$\frac{\partial L}{\partial a_{f0}} = -\frac{3}{8}a_{f0}n_0a_0k_0{}^2\frac{B_1(z_0)}{B_0(z_0)}(t - t_0)\int_{t_0}^{t}\frac{\delta a}{\delta t}\,dt$$

$$\frac{\partial L}{\partial a_{g0}} = \frac{a_{g0}}{a_{f0}}\frac{\partial L}{\partial a_{f0}}, \quad n_0\frac{\partial L}{\partial n_0} = \frac{1}{2}n_0(t - t_0)\frac{\partial n}{\partial n_0}$$

$$\frac{\partial L}{\partial L_0} = \frac{\partial L}{\partial \chi_0} = \frac{\partial L}{\partial \psi_0} = 0$$

Partials of element χ:

$$\frac{\partial \chi}{\partial a_{f0}} = \frac{\partial \chi}{\partial a_{g0}} = \frac{\partial \chi}{\partial n_0} = \frac{\partial \chi}{\partial L_0} = \frac{\partial \chi}{\partial \chi_0} = \frac{\partial \chi}{\partial \psi_0} = 0$$

Partials of element ψ:

$$\frac{\partial \psi}{\partial a_{f0}} = \frac{\partial \psi}{\partial a_{g0}} = \frac{\partial \psi}{\partial n_0} = \frac{\partial \psi}{\partial L_0} = \frac{\partial \psi}{\partial \chi_0} = \frac{\partial \psi}{\partial \psi_0} = 0$$

Table 2 Circumferential partials including bulge

Time (min from epoch)	Element a_f Anal.	Element a_f Var.	Element a_f Num.	Element a_g Anal.	Element a_g Var.	Element a_g Num.	Mean motion n Anal	Mean motion n Var.	Mean motion n Num.
1400	−1.7679	−1.7682	−1.7683	−1.2938	−1.3013	−1.3023	100.1920	100.2085	100.2110
1410	−0.4788	−0.4810	−0.4809	−1.8643	−1.8676	−1.8687	101.3205	101.1996	101.2053
1420	0.8947	0.8898	0.8902	−1.5252	−1.5269	−1.5281	101.7761	101.7518	101.7595
1430	1.7727	1.7701	1.7706	−0.4270	−0.4262	−0.4271	101.7914	101.9517	101.9582
1440	1.8099	1.8087	1.8090	0.9659	0.9797	0.9701	101.8985	102.0577	102.0639
1450	1.0100	1.0085	1.0088	2.0973	2.1002	2.0996	102.3312	102.3144	102.3195
1460	−0.3042	−0.3026	−0.3020	2.5255	2.5240	2.5229	102.9750	102.8576	102.8678
1470	−1.6127	−1.6097	−1.6088	2.0837	2.0814	2.0799	103.7342	103.7303	103.7343
1480	−2.3987	−2.3991	−2.3986	0.9461	0.9422	0.9406	104.7338	104.9035	104.9098
1490	−2.3460	−2.3474	−2.3473	−0.4417	−0.4495	−0.4507	106.0908	106.2442	106.2537
1500	−1.4564	−1.4566	−1.4568	−1.5232	−1.5296	−1.5307	107.5375	107.4946	107.5136
9950	−2.7323	−2.7291	−2.7287	0.2409	0.2346	0.2264	714.5246	713.7740	713.8222
9960	−1.3171	−1.3182	−1.3175	0.2309	0.2285	0.2215	714.9285	714.4587	714.5030
9970	−0.1842	−0.1881	−0.1866	1.0722	1.0757	1.0698	711.4233	712.3014	712.3506
9980	0.2012	0.1975	0.2000	2.4062	2.4155	2.4102	707.5964	708.8563	708.9046
9990	−0.2983	−0.3009	−0.2975	3.6921	3.6985	3.6930	705.7381	705.9270	705.9773
10000	−1.4704	−1.4690	−1.4648	4.4206	4.4193	4.4124	705.4496	704.6830	704.7409
10010	−2.8460	−2.8428	−2.8384	4.3043	4.3020	4.2934	705.9307	705.6365	705.6797
10020	−3.8837	−3.8853	−3.8816	3.3929	3.3927	3.3829	707.8573	708.8131	708.8600
10030	−4.1765	−4.1799	−4.1778	2.0471	2.0434	2.0336	712.3594	713.5816	713.6253
10040	−3.5933	−3.5915	−3.5907	0.7994	0.7920	0.7830	718.3609	718.4032	718.4563
10050	−2.3466	−2.3441	−2.3437	0.1652	0.1598	0.1519	722.1451	721.2751	721.3215

Time (min from epoch)	Mean longitude L Anal.	Mean longitude L Var.	Mean longitude L Num.	Element χ Anal.	Element χ Var.	Element χ Num.	Element ψ Anal.	Element ψ Var.	Element ψ Num.
1400	1.2305	1.2255	1.2253	−1.1649	−1.1589	−1.1606	−0.3733	−0.3723	−0.3712
1410	1.2364	1.2306	1.2304	−1.1654	−1.1611	−1.1623	−0.3750	−0.3756	−0.3750
1420	1.2341	1.2313	1.2311	−1.1675	−1.1653	−1.1659	−0.3777	−0.3798	−0.3799
1430	1.2266	1.2280	1.2277	−1.1716	−1.1711	−1.1719	−0.3818	−0.3852	−0.3851
1440	1.2202	1.2230	1.2227	−1.1774	−1.1780	−1.1795	−0.3868	−0.3906	−0.3898
1450	1.2178	1.2187	1.2184	−1.1833	−1.1839	−1.1856	−0.3914	−0.3942	−0.3930
1460	1.2180	1.2167	1.2164	−1.1866	−1.1858	−1.1870	−0.3944	−0.3951	−0.3945
1470	1.2195	1.2177	1.2174	−1.1863	−1.1830	−1.1836	−0.3961	−0.3947	−0.3947
1480	1.2238	1.2221	1.2218	−1.1835	−1.1780	−1.1789	−0.3971	−0.3947	−0.3946
1490	1.2323	1.2291	1.2288	−1.1809	−1.1746	−1.1762	−0.3981	−0.3963	−0.3955
1500	1.2416	1.2361	1.2358	−1.1802	−1.1746	−1.1764	−0.3994	−0.3990	−0.3979
9950	2.0063	1.9962	1.9943	−2.5206	−2.5148	−2.5231	−2.6359	−2.6448	−2.6404
9960	2.0063	1.9986	1.9966	−2.5214	−2.5181	−2.5256	−2.6378	−2.6482	−2.6445
9970	1.9954	1.9934	1.9915	−2.5166	−2.5177	−2.5252	−2.6474	−2.6590	−2.6562
9980	1.9835	1.9844	1.9825	−2.5134	−2.5167	−2.5248	−2.6631	−2.6755	−2.6726
9990	1.9772	1.9758	1.9739	−2.5148	−2.5165	−2.5251	−2.6797	−2.6918	−2.6879
10000	1.9754	1.9706	1.9687	−2.5173	−2.5164	−2.5245	−2.6902	−2.7007	−2.6963
10010	1.9756	1.9706	1.9687	−2.5178	−2.5155	−2.5230	−2.6909	−2.6990	−2.6954
10020	1.9799	1.9764	1.9745	−2.5180	−2.5156	−2.5231	−2.6833	−2.6898	−2.6871
10030	1.9914	1.9870	1.9851	−2.5224	−2.5191	−2.5273	−2.6727	−2.6792	−2.6763
10040	2.0071	1.9988	1.9969	−2.5310	−2.5257	−2.5343	−2.6644	−2.6722	−2.6682
10050	2.0165	2.0065	2.0046	−2.5371	−2.5316	−2.5397	−2.6614	−2.6708	−2.6665

Correction of Model Parameters

If the orbit correction includes model parameters in addition to the six orbital elements, the G_2 matrix is expanded by the appropriate columns of partial derivatives. These columns are represented by the $G_2{}^m$ matrix in $G_2 = G_2{}^0 \vdots G_2{}^m$;

$$G_2{}^m = G_2{}^m(\text{drag}) \vdots G_2{}^m(\text{thrust}) \ldots$$

where $G_2{}^m$(drag) represents a six-element column vector of the partial derivatives of the current orbit elements with respect to the drag parameter $B = C_DA/m$, and similar columns for the thrust and other model parameters.

The partial derivatives of the elements with respect to the ballistic parameter $B = C_DA/m$ are:

$$B\frac{\partial a_f}{\partial B} = a_{f0}\int_{t_0}^{t}\frac{1}{e}\frac{\delta e}{\delta t}\,dt,\; B\frac{\partial a_g}{\partial B} = a_{g0}\int_{t_0}^{t}\frac{1}{e}\frac{\delta e}{\delta t}\,dt$$

$$B\frac{\partial n}{\partial B} = -\frac{3}{2}\frac{n_0}{a_0}\int_{t_0}^{t}\frac{\delta a}{\delta t}\,dt$$

$$B\frac{\partial L}{\partial B} = -\frac{3}{4}\frac{n_0}{a_0}(t - t_0)\int_{t_0}^{t}\frac{\delta a}{\delta t}\,dt,\; B\frac{\partial \chi}{\partial B} = B\frac{\partial \psi}{\partial B} = 0$$

The partials used for correcting duration thrust parameters, impulsive thrust parameters, outgassing constants, the solar radiation pressure parameter and station coordinates are presented in Ref. 5.

Discontinuities in the Force Model

Satellites can be affected by discontinuous force models. These discontinuities occur when an impulse is applied, when a duration orbit adjust is started or stopped, when outgassing starts or stops, and when the drag factor is changed. It is necessary to consider what happens to the G_2 matrices at these discontinuities.

Up to now only part of the G_2 matrix has been considered. Both the matrix of partials of instantaneous orbit parameters with respect to epoch orbit parameters, $G_2{}^0 = [\partial p_j{}^0/\partial p_{k0}{}^0]$, and the partials of the instantaneous orbit parameters with

Table 3 Circumferential partials including drag

Time (min from epoch)	Element a_f			Element a_g			Mean motion n		
	Anal.	Var.	Num.	Anal.	Var.	Num.	Anal.	Var.	Num.
1400	−0.7873	−0.9981	−0.9186	0.2732	0.2672	0.3943	103.8038	103.8572	103.8025
1410	−1.6039	−1.8208	−1.7400	−0.9211	−0.9126	−0.7838	105.1093	105.1486	105.0985
1420	−1.4680	−1.6921	−1.6109	−2.3801	−2.3493	−2.2202	105.9960	106.0123	105.9637
1430	−0.4427	−0.6693	−0.5878	−3.3953	−3.3564	−3.2279	106.3766	106.3840	106.3325
1440	0.9733	0.7467	0.8288	−3.5028	−3.4667	−3.3386	106.4390	106.4503	106.3948
1450	2.1117	1.8851	1.9680	−2.6866	−2.6605	−2.5311	106.5166	106.5400	106.4853
1460	2.4618	2.2337	2.3175	−1.3522	−1.3376	−1.2049	106.9167	106.9547	106.9005
1470	1.8775	1.6462	1.7317	−0.1153	−0.1113	0.0259	107.7930	107.8450	107.7828
1480	0.6211	0.3861	0.4740	0.4558	0.4525	0.5937	109.0954	109.1581	109.0901
1490	−0.7370	−0.9761	−0.8859	0.0738	0.7000	0.2143	110.5859	110.6512	110.5839
1500	−1.5501	−1.7958	−1.7042	−1.1239	−1.1122	−0.9661	111.9273	111.9778	111.9178
9950	21.5551	8.7877	12.9101	−86.7984	−79.1665	−70.7190	810.8931	842.6935	813.5122
9960	21.7571	8.9247	13.0657	−88.4754	−80.6599	−72.2066	812.7605	844.4332	815.2896
9970	22.6842	9.8533	13.9917	−89.3670	−81.4403	−73.0409	811.0789	842.5436	813.5372
9980	23.9036	11.1291	15.2490	−89.1346	−81.2230	−72.9004	807.1336	838.4703	809.5818
9990	24.8659	12.1558	16.2541	−88.0229	−80.2127	−71.9429	803.2758	834.6581	805.7850
10000	25.1635	12.4778	16.5649	−86.6690	−78.9673	−70.6931	801.6876	833.2951	804.2753
10010	24.6884	11.9678	16.0630	−85.7623	−78.1303	−69.7881	803.4409	835.4058	806.1059
10020	23.6720	10.8658	14.9888	−85.7844	−78.1633	−69.7108	808.1540	840.5110	810.8946
10030	22.6016	9.6837	13.8454	−86.8373	−79.1483	−70.5830	814.1405	846.7894	816.9001
10040	22.0176	8.9924	13.1892	−88.5679	−80.7200	−72.0830	819.0338	851.7588	821.7489
10050	22.2363	9.1465	13.3618	−90.2397	−82.2066	−73.5650	820.8437	853.4374	823.4670

Time (min from epoch)	Mean longitude L			Element χ			Element ψ		
	Anal.	Var.	Num.	Anal.	Var.	Num.	Anal.	Var.	Num.
1400	1.0462	1.0487	1.0464	0.4677	0.4669	0.4651	−0.8101	−0.8108	−0.8099
1410	1.0518	1.0542	1.0519	0.4653	0.4644	0.4625	−0.8059	−0.8066	−0.8056
1420	1.0530	1.0554	1.0532	0.4647	0.4639	0.4619	−0.8049	−0.8056	−0.8046
1430	1.0494	1.0518	1.0495	0.4663	0.4655	0.4635	−0.8077	−0.8084	−0.8074
1440	1.0426	1.0450	1.0428	0.4693	0.4685	0.4666	−0.8129	−0.8136	−0.8127
1450	1.0361	1.0386	1.0363	0.4723	0.4714	0.4695	−0.8180	−0.8188	−0.8179
1460	1.0327	1.0352	1.0329	0.4738	0.4730	0.4709	−0.8207	−0.8215	−0.8204
1470	1.0339	1.0365	1.0341	0.4733	0.4724	0.4702	−0.8198	−0.8205	−0.8194
1480	1.0392	1.0419	1.0393	0.4709	0.4700	0.4678	−0.8156	−0.8164	−0.8153
1490	1.0463	1.0490	1.0464	0.4677	0.4668	0.4646	−0.8101	−0.8109	−0.8098
1500	1.0518	1.0545	1.0519	0.4653	0.4643	0.4621	−0.8058	−0.8067	−0.8056
9950	1.0498	1.1718	1.0508	0.4650	0.4184	0.3051	−0.8055	−0.8229	−0.7883
9960	1.0511	1.1731	1.0521	0.4645	0.4178	0.3042	−0.8045	−0.8276	−0.7873
9970	1.0478	1.1693	1.0488	0.4659	0.4193	0.3061	−0.8070	−0.8295	−0.7900
9980	1.0416	1.1625	1.0426	0.4687	0.4223	0.3097	−0.8118	−0.8279	−0.7950
9990	1.0355	1.1561	1.0365	0.4715	0.4253	0.3129	−0.8166	−0.8235	−0.7998
10000	1.0322	1.1531	1.0332	0.4730	0.4268	0.3142	−0.8192	−0.8183	−0.8021
10010	1.0332	1.1549	1.0343	0.4725	0.4261	0.3129	−0.8184	−0.8148	−0.8010
10020	1.0381	1.1609	1.0392	0.4703	0.4235	0.3095	−0.8145	−0.8147	−0.7971
10030	1.0447	1.1685	1.0457	0.4673	0.4201	0.3053	−0.8094	−0.8181	−0.7920
10040	1.0498	1.1742	1.0509	0.4650	0.4175	0.3021	−0.8054	−0.8233	−0.7880
10050	1.0510	1.1753	1.0521	0.4645	0.4169	0.3012	−0.8045	−0.8279	−0.7871

respect to the model parameters, $G_2{}^m = [\partial p_j{}^0/\partial p_{k0}{}^m]$, have been defined.

Since the model parameters are the same as at epoch, $[\partial p_j{}^m/\partial p_{k0}{}^m] = I$, the identity matrix. Nor do these model parameters depend on the orbit parameters, so $[\partial p_j{}^m/\partial p_{k0}{}^0] = \varphi$, the null matrix.

Therefore, the G_2 matrix in full is

$$\left[\begin{array}{c|c} G_2{}^0 & G_2{}^m \\ \hline \varphi & I \end{array}\right]$$

This now relates the current parameters to those at the last epoch. At the first discontinuity, let $G_2{}^0 = G_x{}^0$ and $G_2{}^m = G_x{}^m$. At a time just past this first discontinuity, the partial derivatives with respect to the original epoch parameters are

$$\left[\begin{array}{c|c} G_2{}^0 & G_2{}^m \\ \hline \varphi & I \end{array}\right]\left[\begin{array}{c|c} G_x{}^0 & G_x{}^m \\ \hline \varphi & I \end{array}\right] = \left[\begin{array}{c|c} G_2{}^0G_x{}^0 & G_2{}^0G_x{}^m + G_2{}^m \\ \hline \varphi & I \end{array}\right]$$

The left-hand G_2 matrix is calculated by the formulas of the preceding sections with the understanding that t_0 means the latest epoch. If the current time is an observation time, use $G_2{}^0G_x{}^0$ and $G_2{}^0G_x{}^m + G_2{}^m$ instead of the $G_2{}^0$ and $G_2{}^m$ indicated in the Introduction. If the current time is another discontinuity, update by letting $G_x{}^0 = G_2{}^0G_x{}^0$ and $G_x{}^m = G_2{}^0G_x{}^m + G_2{}^m$. Obviously, this process is initialized at the original (correction) epoch by setting $G_x{}^0 = I$ and $G_x{}^m = \varphi$. The dimensions of $G_x{}^m$ depend on the number of model parameters solved for.

III. Validation Tests

In order to demonstrate the validity of the analytic partials, they were compared to partials obtained by integrating variational equations and by differencing variant orbits. The variational equations require that the partial derivatives be numerically integrated along the orbit, starting from specific initial conditions at epoch. The variant representations involve varying one nominal orbit element (or one model parameter) at a time and noting the position displacements that result in each case. Obviously, a limitation of the variant orbit technique is the nonlinearity introduced into the components of $\Delta\mathbf{r}$, if $\Delta\mathbf{r}$ represents a displacement in satellite position greater than, say, 0.1 rad. The forced variations in the elements were selected such that they led to a Δr of about 10 naut miles after one day.

Reference 7 describes the basic program used to generate the analytic partials. The initialization of the bulge partials was modified so that the definition of the orbital elements to be corrected would agree exactly with those defined by the variational equations and the variant orbits.

The analytic partials were derived on the basis of six independent orbital elements to be improved by a linear differential correction process. Changing any one of these orbital elements at epoch instantaneously alters the disturbing gravitational function. Unless the value of the semimajor axis is changed accordingly, there will occur a change in the total energy. Thus, in the program[7] the osculating mean motion (which replaces the semimajor axis) is modified following the differential correction of the epoch elements where provision is made for only the second zonal harmonic in the perturbed gravitational potential. Also, the only element that needs to be considered is the angular mean motion because it appears as the cofactor of the time elapsed since epoch in the mean longitude equation. For numerical partial derivatives to represent relationships between position and velocity and six independent epoch orbital parameters requires that, at least, the perturbations in epoch mean motion caused by the forced variation in one of the elements be subtracted from the given value of n before initializing the variant orbit.

Since the numerical partials to be presented were obtained by varying only one element at a time, therefore letting the initial mean motion be dependent on the variations of the other parameters, it was necessary to modify the initialization of the mean motion as it occurs in the mean longitude equation of the analytic partials. The additions were entered through the second zonal harmonic perturbations in n_0 into the partials of mean longitude L of the $G_2{}^0$ matrix;

$$\partial L/\partial p_{k0} = (\partial n_0/\partial p_{k0})(t - t_0)$$

The quantities $\partial n_0/\partial p_{k0}$ follow for each of the six orbital elements p_{k0},

$$\frac{\partial n_0}{\partial a_{f0}} = -3K\left\{\left(\frac{a^3}{r^4}\right)_0 [AG_1{}^0(1,1) + \sin^2 i_0 \sin 2u_0 G_1{}^0(2,1)] + \left(1 - \frac{3}{2}\sin^2 i_0\right)(1 - e_0{}^2)^{-5/2} a_{f0}\right\}$$

$$\frac{\partial n_0}{\partial a_{g0}} = -3K\left\{\left(\frac{a^3}{r^4}\right)_0 [AG_1{}^0(1,2) + \sin^2 i_0 \sin 2u_0 G_1{}^0 \times (2,2)] + \left(1 - \frac{3}{2}\sin^2 i_0\right)(1 - e_0{}^2)^{-5/2} a_{g0}\right\}$$

$$n_0 \frac{\partial n_0}{\partial n_0} = \frac{7}{2} K \left\{\frac{2}{3}\left(1 - \frac{3}{2}\sin^2 i_0\right)\left[\left(\frac{a}{r}\right)_0^3 - (1 - e_0{}^2)^{-3/2}\right] + \left(\frac{a}{r}\right)_0^3 \sin^2 i_0 \cos 2u_0\right\}$$

$$\frac{\partial n_0}{\partial L_0} = -3K\left(\frac{a^3}{r^4}\right)_0 [AG_1{}^0(1,4) + \sin^2 i_0 \sin 2u_0 G_1{}^0(2,4)]$$

$$\frac{\partial n_0}{\partial \chi_0} = -3K(1 + \cos i_0)^2 \left\{\chi_0 \cos i_0 \left[\left(\frac{a}{r}\right)_0^3 - (1 - e_0{}^2)^{-3/2}\right] - \left(\frac{a}{r}\right)_0^3 [\chi_0 \cos i_0 \cos 2u_0 + \psi_0 \sin 2u_0]\right\}$$

$$\frac{\partial n_0}{\partial \psi_0} = -3K(1 + \cos i_0)^2 \left\{\psi_0 \cos i_0 \left[\left(\frac{a}{r}\right)_0^3 - (1 - e_0{}^2)^{-3/2}\right] - \left(\frac{a}{r}\right)_0^3 [\psi_0 \cos i_0 \cos 2u_0 - \chi_0 \sin 2u_0]\right\}$$

Table 4 Radial and circumferential partials of $\Delta B/B$, the ballistic coefficient

Time (min from epoch)	Radial			Circumferential		
	Anal.	Var.	Num.	Anal.	Var.	Num.
1400	−0.0003	−0.0003	−0.0003	0.0140	0.0140	0.0140
1410	−0.0002	−0.0002	−0.0002	0.0142	0.0142	0.0142
1420	−0.0001	−0.0001	−0.0001	0.0145	0.0143	0.0143
1430	−0.0000	−0.0000	−0.0000	0.0146	0.0143	0.0143
1440	−0.0001	−0.0001	−0.0001	0.0146	0.0144	0.0144
1450	−0.0002	−0.0002	−0.0002	0.0147	0.0146	0.0146
1460	−0.0003	−0.0003	−0.0003	0.0149	0.0149	0.0149
1470	−0.0004	−0.0004	−0.0004	0.0152	0.0153	0.0152
1480	−0.0004	−0.0004	−0.0004	0.0155	0.0156	0.0156
1490	−0.0003	−0.0003	−0.0003	0.0158	0.0159	0.0159
1500	−0.0002	−0.0002	−0.0002	0.0161	0.0161	0.0161
9950	−0.0041	−0.0045	−0.0073	0.7350	0.7880	0.7734
9960	0.0006	0.0005	−0.0025	0.7371	0.7893	0.7748
9970	0.0043	0.0044	0.0014	0.7360	0.7875	0.7731
9980	0.0053	0.0055	0.0026	0.7329	0.7842	0.7699
9990	0.0032	0.0032	0.0004	0.7301	0.7819	0.7675
10000	−0.0011	−0.0013	−0.0039	0.7297	0.7823	0.7678
10010	−0.0055	−0.0059	−0.0085	0.7325	0.7860	0.7713
10020	−0.0080	−0.0086	−0.0113	0.7377	0.7921	0.7772
10030	−0.0076	−0.0081	−0.0109	0.7437	0.7987	0.7836
10040	−0.0041	−0.0045	−0.0074	0.7486	0.8033	0.7882
10050	0.0007	0.0006	−0.0025	0.7507	0.8046	0.7896

and

$$K = \tfrac{3}{2} J_2 n_0 a_e{}^2 / a_0$$

$$A = (1 - \tfrac{3}{2}\sin^2 i_0) + \tfrac{3}{2}\sin^2 i_0 \cos 2u_0$$

0 subscript stands for epoch values and $G_1{}^0(i,j)$ stands for the epoch value of the ith row and jth column of the G_1 matrix.

Perturbations due to bulge (J_2), drag, thrust, and thrust rate were included in the numerical experimentation. Examination of the G_2 matrix for J_2 demonstrates that only secular terms due to J_2 are required in the analytic partials for bulge. The remaining terms in the Legendre series expansion for the geopotential will not significantly alter the quantitative comparisons after one day because of their much smaller numerical coefficients. Higher-order terms are required with the variational equations. In the validation this problem was avoided by restricting the force model to J_2 in the equations of motion also. The drag studies were carried out with a satellite in a 90-min orbit having an area-to-mass ratio approximately equal to that of the Agena. Thrusting cases involved the application of approximately half the total Agena impulse, applied independently in the radial, circumferential, and binormal directions. The amount of thrust rate applied was approximately one percent of the thrust constant.

Table 5 Demonstration of nonlinearity (a_f partials inclusive of drag)

Time, min	a_f = 0.00025, numeric			a_f = 0.0001, numeric		
	u	v	Δr, km	u	v	Δr, km
1400	−0.3905	−3.0145	19.39	−0.3961	−2.3987	1.55
1410	−0.9812	−1.9980	14.20	−0.9878	−1.3735	1.08
1420	−1.0946	−0.4574	7.57	−1.1019	0.1748	0.71
1430	−0.6757	0.8584	6.97	−0.6831	1.4981	1.05
1440	0.0750	1.3084	8.36	0.0673	1.9560	1.25
1450	0.7964	0.6689	6.63	0.7882	1.3252	0.98
1460	1.1403	−0.7588	8.74	1.1318	−0.0937	0.72
1470	0.9395	−2.2942	15.81	0.9321	−1.6197	1.19
1480	0.2909	−3.2041	20.52	0.2848	−2.5194	1.62
1490	−0.4927	−3.0553	19.74	−0.4986	−2.3606	1.54
1500	−1.0336	−1.9242	13.93	−1.0405	−1.2204	1.02
9950	−2.2287	−40.9260	261.42	−1.4886	−16.7009	10.69
9960	−2.1685	−38.7989	247.85	−1.4623	−14.5377	9.32
9970	−1.4082	−37.2008	237.44	−0.7301	−12.9303	8.26
9980	−0.3098	−36.9293	235.55	0.3588	−12.6598	8.08
9990	0.6031	−38.1490	243.35	1.2866	−13.8702	8.88
10000	0.8933	−40.3161	257.21	1.6117	−16.0017	10.26
10010	0.4195	−42.4375	270.69	1.1788	−18.0565	11.54
10020	−0.5942	−43.5401	277.73	0.1928	−19.0716	12.16
10030	−1.6649	−43.1331	275.31	−0.8768	−18.5771	11.86
10040	−2.2839	−41.4450	264.74	−1.5199	−16.8231	10.77
10050	−2.1588	−39.3168	251.15	−1.4296	−14.6613	9.40

Table 1 summarizes the numerical results. In all cases the partials remained in phase after seven days. This represented a qualitative design goal of the study. The table indicates, for each parameter and perturbation, the maximum errors at one day, expressed as parts in 600. One part in 600 after one day was the quantitative design goal of the study.

Tables 2, 3, and 4 compare, in tabular form, the bulge and drag partials as obtained from the analytic formulas, the variational equations, and the variant representations at one- and seven-day intervals from epoch. Similar studies were carried out for thrust and thrust rate.[8]

Small deviations occur for a few elements in the circumferential components which reflect a nonlinear relationship between the perturbed orbital period and these elements, thereby disqualifying them in a description (and design goal) based upon first-order partials. Those which are clearly nonlinear include the circumferential position component due to variations in eccentricity, in a drag environment, and to radial thrust and radial thrust rate. The latter may be visualized intuitively by considering a circular orbit perturbed by a radial thrust of finite duration. Regardless of the sense of this thrust (radially inward or outward), the period of the orbit is increased and the subsequent circumferential position has a secular component which is an even function of this radial thrust. A description of this phenomenon by a first-order correction theory is not strictly valid and fails to meet the quantitative (1:600) design goal, although the radial displacement does. The nonlinearity in circumferential behavior with variations in a_f and a_g in a drag environment stems from the nonlinear variation in period decay rate with perigee altitude, and is quantitatively demonstrated by Table 5. In this test, a circular 90-min satellite orbit was successfully perturbed (in a_f) by 0.000,10 and 0.000,25. Had the process been linear, the resulting displacement $|\Delta \mathbf{r}|$ would have been proportional to Δa_f and the corresponding partials identical; clearly this linear hypothesis is rejected. Accordingly, the summary Table 1 omits the corresponding data.

Finally, it should be noted that whereas the accuracy of partials derived by numerical integration of the variational equations is generally better after one day, the reverse is generally true over extended periods like a week. This latter fact is believed to be due to the restriction to single precision in the variational equations, leading to significant error accumulation even with the 48-bit word length in the CDC-3600 computer used in this study. No significant difference would be noted in orbit parameter fits to observations over a few days from epoch for a 90-min orbit, other than the significant speed advantage enjoyed by analytic partials over their numerically-integrated counterparts; for routine orbit maintenance, this speed advantage leads to over-all program running time reduction by a factor of three or more. As an example, when 140 observations over 1630 minutes were used in a fit, one iteration using analytic partials took 2.13 min of computer time. One iteration using variational equations consumed 6.37 min.

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